Locomotion of helical bodies in viscoelastic fluids: enhanced swimming at large helical amplitudes
Supplementary material

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In this document we describe in greater detail the numerical method used for the study described in the primary text, and tabulate a selection of rheological measurements and microorganism geometries relevant to our investigation. In §I the filament geometry and the Oldroyd-B equations are written in a helical coordinate system. The numerical discretization and method of solution are then described; numerous differential operations in the helical coordinate system are included in an accompanying appendix. In §II we provide the measured fluid rheologies and biological data from recent experiments.

I. NUMERICAL METHOD

Filament geometry and nondimensionalization

In our investigation the Stokes/Oldroyd-B equations are solved numerically in a cross-sectional plane of an infinitely long, rotating helical filament. A material point on the filament centerline is described in the fixed lab frame by

$$\boldsymbol{x}_0(s,t) = b[\cos(ks+\Omega t)\hat{\boldsymbol{x}} + \sin(ks+\Omega t)\hat{\boldsymbol{y}}] + (\alpha s + U^* t)\,\hat{\boldsymbol{z}}, \tag{1}$$

where b is the helical radius, k is the wavenumber, Ω is a fixed rotation rate, and U^* is the swimming speed. The parameter s is made the arc-length by setting $\alpha = \sqrt{1-(bk)^2}$. The body is shaped such that the boundary in a cross-sectional plane perpendicular to the $\hat{z}$ axis is circular with radius A/k. Scaling velocities on Ω/k, lengths on $1/k$, and time upon $1/\Omega$, the dimensionless filament centerline is given by

$$\boldsymbol{x}_0(s,t) = \beta\,[\cos(s+t)\hat{\boldsymbol{x}} + \sin(s+t)\hat{\boldsymbol{y}}] + (\alpha s + Ut)\,\hat{\boldsymbol{z}}, \tag{2}$$

with $\beta = bk$ (hence $\alpha^2+\beta^2 = 1$) and $U = U^* k/\Omega$, where all variables are understood to be dimensionless. In the main text we described the geometry in terms of the pitch angle, ψ, with $(\alpha,\beta) = (\cos(\psi), \sin(\psi))$.

We now describe a time-dependent, helical coordinate system (r,θ,ζ) that conforms to the helical body as it translates with constant dimensionless speed U and rotates with unit angular velocity through the fluid. Defining the unit vectors $(\hat{\mathbf{x}}(\zeta,t), \hat{\mathbf{y}}(\zeta,t))$, which rotate and translate along with the body, a point in space is written as

$$\boldsymbol{x} = (\beta + r\cos(\theta))\,\hat{\mathbf{x}}(\zeta,t) + r\sin(\theta)\,\hat{\mathbf{y}}(\zeta,t) + (\zeta + Ut)\,\hat{\boldsymbol{z}}, \tag{3}$$

$$\hat{\mathbf{x}}(\zeta,t) = \cos(\zeta/\alpha + t)\hat{\boldsymbol{x}} + \sin(\zeta/\alpha + t)\hat{\boldsymbol{y}}, \tag{4}$$

$$\hat{\mathbf{y}}(\zeta,t) = -\sin(\zeta/\alpha + t)\hat{\boldsymbol{x}} + \cos(\zeta/\alpha + t)\hat{\boldsymbol{y}}, \tag{5}$$

where r is the (dimensionless) distance from the filament centerline in the plane perpendicular to $\hat{\boldsymbol{z}}$ (a plane of constant ζ). Holding (r,θ) fixed, the line parameterized by ζ is helical and periodic on $\zeta \in [0, 2\pi\alpha]$; holding $r = A$ fixed, the material filament surface is parameterized by (ζ,θ). A polar coordinate system (r,θ) is used to parameterized space in a cross-sectional plane of the helical body, where

$$\hat{\boldsymbol{r}} = \cos(\theta)\hat{\mathbf{x}} + \sin(\theta)\hat{\mathbf{y}}, \quad \hat{\boldsymbol{\theta}} = -\sin(\theta)\hat{\mathbf{x}} + \cos(\theta)\hat{\mathbf{y}}. \tag{6}$$

The velocity boundary conditions for rigid body motion (applying the no-slip condition) are then written as

$$\boldsymbol{u}(r = A,\theta,\zeta) = -A\sin(\theta)\,\hat{\mathbf{x}} + (\beta + A\cos(\theta))\,\hat{\mathbf{y}} + U\hat{\boldsymbol{z}} = \beta\sin(\theta)\hat{\boldsymbol{r}} + (A + \beta\cos(\theta))\hat{\boldsymbol{\theta}} + U\hat{\boldsymbol{z}}, \tag{7}$$

with the ζ dependence implicit in the definitions of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$, while the passage of a helical wave requires instead that

$$\boldsymbol{u}(r = A,\theta,\zeta) = \beta\,\hat{\mathbf{y}} + U\hat{\boldsymbol{z}} = \beta\sin(\theta)\hat{\boldsymbol{r}} + \beta\cos(\theta)\hat{\boldsymbol{\theta}} + U\hat{\boldsymbol{z}}. \tag{8}$$

The boundary conditions differ by an extra rotation about the filament centerline in the former, with diminishing consequences for decreasing A/β.

Oldroyd-B equations in the helical coordinate system

Given that the helical filament is of infinite length, the steady-state (with respect to the body frame) is instantaneously described by the velocity and stress fields in any cross-sectional plane perpendicular to $\hat{\boldsymbol{z}}$. As described in the main text, in the fluid domain we solve the Oldroyd-B equations in the Stokes limit,

$$\nabla p = \nabla\cdot\boldsymbol{\tau}, \tag{9}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{10}$$

where $\boldsymbol{u}$ is the fluid velocity, p is the pressure, and $\boldsymbol{\tau}$ is the total (deviatoric) viscoelastic stress tensor, $\boldsymbol{\tau} = \eta_s\dot{\boldsymbol{\gamma}} + \boldsymbol{\tau}_p$. Here η_s is the solvent viscosity, $\dot{\gamma} = \nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T$ is the rate-of-strain tensor, and $\boldsymbol{\tau}_p$ is the extra stress contributed by the polymer suspension. In the Newtonian case, where $\boldsymbol{\tau}_p = 0$, the divergence of the stress reduces to $\nabla\cdot\boldsymbol{\tau} = \eta_s\Delta\boldsymbol{u}$. In the Oldroyd-B model, the polymer stress is given by

$$\boldsymbol{\tau}_p + \lambda_1\overset{\triangledown}{\boldsymbol{\tau}}_p = \eta_p\dot{\boldsymbol{\gamma}}, \tag{11}$$

where η_p is the polymer suspension viscosity, $\eta = \eta_s + \eta_p$ is the total viscosity, and λ_1 is the polymer relaxation time-scale [1, 2]. The upper convective time derivative is defined by

$$\overset{\triangledown}{\boldsymbol{\tau}} = \boldsymbol{\tau}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{\tau} - \nabla\boldsymbol{u}^T\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\nabla\boldsymbol{u}. \tag{12}$$

Manipulation of Eqn. (11) leads to an equation for the total stress, $\boldsymbol{\tau}$,

$$\boldsymbol{\tau} + \lambda_1\overset{\triangledown}{\boldsymbol{\tau}} = \eta\left(\dot{\boldsymbol{\gamma}} + \lambda_2\overset{\triangledown}{\dot{\boldsymbol{\gamma}}}\right), \tag{13}$$

with $\lambda_2 = (\eta_s/\eta)\lambda_1$. Scaling time on $1/\Omega$, velocities on Ω/k and stresses on $\eta\,\Omega$, we find the dimensionless constitutive relation as stated in the main text,

$$\boldsymbol{\tau} + \mathrm{De}\,\overset{\triangledown}{\boldsymbol{\tau}} = \dot{\boldsymbol{\gamma}} + (\eta_s/\eta)\mathrm{De}\,\overset{\triangledown}{\dot{\boldsymbol{\gamma}}}, \tag{14}$$

and $\boldsymbol{\tau} = (\eta_s/\eta)\dot{\boldsymbol{\gamma}} + \boldsymbol{\tau}_p$. The Deborah number compares the rate of rotation to the rate of elastic relaxation in the fluid, $\mathrm{De} = \lambda_1\Omega$. Once again, all variables are understood to be dimensionless in the above. The dimensionless momentum balance equation (9) is then given by

$$\nabla p = (\eta_s/\eta)\boldsymbol{\Delta u} + \nabla\cdot\boldsymbol{\tau}_p, \tag{15}$$

Note that in the limit of $\mathrm{De}\to 0$, we have $\boldsymbol{\tau}_p = (1-(\eta_s/\eta))\dot{\boldsymbol{\gamma}}$, $\boldsymbol{\tau} = \dot{\boldsymbol{\gamma}}$, and $\nabla p = \nabla\cdot\boldsymbol{\tau} = \Delta\boldsymbol{u}$ as expected. Finally, the dimensionless fluid force per wavelength acting axially (in the $\hat{\boldsymbol{z}}$ direction) at any time or station in z is given by

$$F_z = 2\pi\alpha A\int_0^{2\pi}\Big[(\beta/\alpha)\sin(\theta)(p-\tau_{zz}) + \tau_{rz}\Big]_{r=A}\,d\theta. \tag{16}$$

Exploiting helical symmetry

Under the transformation to helical coordinate system shown in (3), we have

$$\frac{\partial}{\partial \mathsf{x}} = \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial\theta}, \tag{17}$$

$$\frac{\partial}{\partial \mathsf{y}} = \sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial\theta}, \tag{18}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial\zeta} - \frac{1}{\alpha}\left(\left[1+\frac{\beta\cos(\theta)}{r}\right]\frac{\partial}{\partial\theta} + \beta\sin(\theta)\frac{\partial}{\partial r}\right). \tag{19}$$

Consider a steady-state with respect to the body frame. Invoking helical symmetry, for any scalar quantity Φ we have that $\partial\Phi/\partial\zeta = 0$. Equation (19) then shows how an axial (z) derivative may be written solely in terms of r and θ derivatives. Hence, it is straightforward to show that the del operator may be written as

$$\nabla = \hat{\mathbf{x}}\partial_{\mathsf{x}} + \hat{\mathbf{y}}\partial_{\mathsf{y}} + \hat{\boldsymbol{z}}\partial_z = \hat{\boldsymbol{r}}\partial_r + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta + \hat{\boldsymbol{z}}\left[-\frac{1}{\alpha}\left(\left[1+\frac{\beta\cos(\theta)}{r}\right]\partial_\theta + \beta\sin(\theta)\partial_r\right)\right], \tag{20}$$

(see Appendix A). The gradient of any scalar variable such as the pressure is then given simply by

$$\nabla p = \hat{\boldsymbol{r}}\frac{\partial p}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial p}{\partial\theta} + \hat{\boldsymbol{z}}\frac{\partial p}{\partial z}, \tag{21}$$

where

$$\frac{\partial p}{\partial z} = -\frac{1}{\alpha}\left(\left[1+\frac{\beta\cos(\theta)}{r}\right]\frac{\partial p}{\partial\theta} + \beta\sin(\theta)\frac{\partial p}{\partial r}\right). \tag{22}$$

Similarly, the divergence of a vector field $\boldsymbol{u} = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{z}}$ is given by

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u}{\partial r} + \frac{1}{r}\left(\frac{\partial v}{\partial r} + u\right) + \frac{\partial w}{\partial z}, \tag{23}$$

where again $\partial w/\partial z$ is written in terms of only r and θ derivatives. Numerous differential operations in the helical coordinate system are included in Appendix A.

A final simplification comes with the consideration of time-derivatives. For this simplification, we choose to solve the equations of motion in a translating plane of constant $z' = z + (\alpha - U)t$, or equivalently for constant $\zeta' = \zeta + \alpha t$, in which the filament geometry (though not material) appears to be fixed in space. The equations of motion go unchanged in the translating frame as a consequence of Galilean invariance. Assuming a steady state of flow and stress relative to the body geometry we have for any scalar function $\Phi(\boldsymbol{x}, t)$ that

$$0 = \frac{d}{dt}\Phi(\boldsymbol{x}, t)\Big|_{r,\theta,\zeta'} = \Phi_t + \boldsymbol{x}_t \cdot \nabla\Phi\Big|_{r,\theta,\zeta'}, \tag{24}$$

with subscripts denoting partial derivatives. A partial time derivative may therefore be written in terms of spatial derivatives confined to a plane of constant ζ',

$$\Phi_t\Big|_{r,\theta,\zeta'} = (\alpha - U)\Phi_z\Big|_{r,\theta,\zeta'} = \left(\frac{U}{\alpha} - 1\right)\left(\left[1 + \frac{\beta\cos(\theta)}{r}\right]\Phi_\theta + \beta\sin(\theta)\Phi_r\right)\Big|_{r,\theta,\zeta'}. \tag{25}$$

Discretization and method of solution

The Oldroyd-B equations in the periodic steady-state are solved as just described using a mixed pseudo-spectral / finite differences approach. To begin, all variables are described in a Fourier basis in the azimuthal angle θ,

$$p(r,\theta) = \sum_{k=-N}^{N} \hat{p}_k(r)e^{ik\theta}, \quad \boldsymbol{u}(r,\theta) = \sum_{k=-N}^{N} \hat{\boldsymbol{u}}_k(r)e^{ik\theta}, \quad \boldsymbol{\tau}(r,\theta) = \sum_{k=-N}^{N} \hat{\boldsymbol{\tau}}_k(r)e^{ik\theta}. \tag{26}$$

The radial distance from the filament center, r, is discretized as $r_m = A + (A_{out} - A)((m-1)/M)^2$ for $m = 1, 2, ..., M + 1$. Here A_{out} is an outer radial distance taken large enough so that the numerical results are not altered significantly with any further increases in its value. For example, in the helical wave case with $A = 0.5$ and $\psi = \pi/40$ from the main text, it is sufficient to take $A_{out} = 20A$ in order that the errors be due to discretization alone. Moreover, the swimming speed computed using only $M = 200$ and $N = 2$ changes by less than 0.2% with any increases in M or N. For larger pitch angles such as $\psi = \pi/5$, again with $A = 0.5$, we require $A_{out} = 30A$, $M = 200$, and $N = 6$ for comparable precision, with resolution requirements increasing with the Deborah number, and in the study of rigid body rotation. The computational costs required to accurately study bodies of even larger pitch angle were generally found to be prohibitive in the current approach.

The choice of a quadratic scaling in the discretization of r increases the resolution of the fluid velocity and stress near the body where it is most needed. Radial derivatives are computed using fourth-order finite difference formulae derived specifically for the spatially varying grid spacing in r. Formulae for derivatives near the endpoints are skewed, depending upon variable information on the boundary and inside the domain of computation. Azimuthal derivatives, meanwhile, are computed in Fourier space using the definitions in (26).

Iterative scheme

Our approach is to develop an iterative numerical scheme. Initial guesses for the velocity and pressure fields, and the swimming speed, are relaxed until the equations of motion are satisfied. As unknowns we take the Fourier modes (with radial dependence) describing the velocity field $\hat{\boldsymbol{u}}_k(r_m)$ and the pressure $\hat{p}_k(r_m)$, and also the swimming speed U, which are all formed into a single vector $\boldsymbol{v}$ of length $4M(2N+1)+1$. For a given approximation to the velocity field $\boldsymbol{u}$ and swimming speed U, the stress $\boldsymbol{\tau}$ is treated as an auxiliary variable and is determined immediately through the constitutive equation (14). This step is performed rapidly using a Generalized Minimal Residual iterative scheme (GMRES) [3].

The momentum balance equation (9) is required to be satisfied at every radial grid point (collocation), and when integrated against the Fourier basis functions (a Galerkin scheme) [4],

$$\int_0^{2\pi} e^{iK\theta} \left[\nabla p - \nabla \cdot \boldsymbol{\tau}\right](r_m, \theta)\, d\theta = 0, \tag{27}$$

for $m = 1, 2..., M+1$, and for $K = -N, -N+1, ..., N-1, N$. The momentum balance equation is replaced by the no-slip boundary conditions for the velocity field on the innermost and outermost radial gridpoints. Simultaneously we require that

$$\int_0^{2\pi} e^{iK\theta} \left[\boldsymbol{\Delta} p - \nabla \cdot (\nabla \cdot \boldsymbol{\tau}_p)\right](r_m, \theta)\, d\theta = 0, \tag{28}$$

again for each radial gridpoint and against each Fourier basis function. To derive the above we have used the mass conservation equation $\nabla \cdot \boldsymbol{u} = 0$. This equation is replaced by the following boundary conditions at the innermost and outermost radial boundaries,

$$\left.\frac{\partial p}{\partial r}\right|_{r=A} = -(\eta_s/\eta)\, \hat{\boldsymbol{r}} \cdot (\nabla \times \nabla \times \boldsymbol{u}) + \hat{\boldsymbol{r}} \cdot (\nabla \cdot \boldsymbol{\tau}_p)\Big|_{r=A}, \tag{29}$$

$$\left.\frac{1}{r}\frac{\partial p}{\partial \theta}\right|_{r=A_{out}} = -(\eta_s/\eta)\, \hat{\boldsymbol{\theta}} \cdot (\nabla \times \nabla \times \boldsymbol{u}) + \hat{\boldsymbol{\theta}} \cdot (\nabla \cdot \boldsymbol{\tau}_p)\Big|_{r=A_{out}}. \tag{30}$$

Here we are free to assert that the pressure on the outer boundary has mean zero, thereby fixing the constant of integration (which has no bearing on the flow velocity or swimming speed). The condition that the velocity field is divergence free is retained as a diagnostic tool and is verified at the end of each computation.

Finally, we require the net force on the swimming body to be zero, $F_z = 0$ (see (16)). The equations and constraints above are written together as a large nonlinear system of $4M(2N+1)+1$ equations, $\mathcal{F}[\boldsymbol{v}] = \boldsymbol{0}$. The solution to this system of equations is found by the application of Broyden's method, an adaptation of Newton's method that updates the Jacobian matrix $\boldsymbol{J} = [\partial\mathcal{F}/\partial\boldsymbol{v}]$ at each iteration without recomputing each matrix entry [5]. For the first iteration, the Jacobian is computed using a centered finite difference approximation. $\boldsymbol{J}$ is sparse and fast algorithms for matrix inversion and multiplication are employed. Since each column of $\boldsymbol{J}$ can be determined independent of all other columns, this step is performed in blocks on a parallel architecture without complication. Close initial guesses $\boldsymbol{v}^0$ for a given Deborah number are provided by extrapolation from previous computations for smaller De, starting with the simplest computation, that of Stokes flow (De = 0).

To validate the code we have tested against an exact solution of cylindrical helical flow in an annulus, compared to other computations of Newtonian flows, and performed convergence studies in the general setting, all of which indicate the expected numerical accuracy. Extensive details will be provided in a separate manuscript.

II. RHEOLOGICAL AND BIOLOGICAL DATA

We include as Table I a selection of recent experimental investigations of fluid rheological measurements and microorganism geometries. Detailed fluid rheology and microorganism geometries are not generally reported simultaneously in the literature, making a full comparison to the swimming speeds observed in vivo quite challenging. The data shown, however, is generally consistent with our results, in that swimming in viscoelastic fluids appears to be enhanced for slender filaments of large amplitude (e.g., pitch angle $\psi \gtrsim 30^\circ$ for helical bodies). The organism geometry is reported in Table I as the dimensionless filament radius A. Whether the organism swims faster or slower in a viscoelastic medium than in a Newtonian fluid is indicated as 'relative motility'.

TABLE I. Biological data (*denotes peritrichous cells)

Organism	Kinematics	Filament radius, A	Medium	Relative motility
B. burgdorferi	flat wave, $\psi = 35^\circ$	0.5 [6]	gelatin	faster (no slip) [7]
Mouse sperm (fresh)	small amplitude	0.03 [8, 9]	viscoelastic	slower [10]
Mouse sperm (hyperactivated)	large amplitude	0.03 [8, 9]	viscoelastic	faster [10]
E. coli *	helical, $\psi = 30^\circ$	0.03 [11]	Methocel (gel)	faster [12]
P. aeruginosa	$\psi = 40^\circ$ [13]	0.03	mucin	faster [14]
H. pylori *	/	0.03	PGM† gel (De $\sim 10^3$)	immotile [15]
H. pylori *	helical, $\psi = 57^\circ$	0.03	PGM† solution (De ~ 1)	motile [15]
C. elegans	planar undulation	0.3	CMC†† gel (De ~ 1)	slower [16]

† Porcine gastric mucin †† Carboxy-methyl cellulose solution

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APPENDIX A. DIFFERENTIAL OPERATIONS IN THE HELICAL COORDINATE SYSTEM

From the definition of the helical coordinate system in (3), we now derive a number of differential operators acting on scalars, vectors, and tensors that appear in the equations of motion. Beginning with the relations in (6) and (19) we have that

$$\frac{\partial}{\partial r}\hat{\boldsymbol{r}} = 0, \quad \frac{\partial}{\partial r}\hat{\boldsymbol{\theta}} = 0, \tag{31}$$

$$\frac{\partial}{\partial \theta}\hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{r}}, \tag{32}$$

$$\frac{\partial}{\partial z}\hat{\boldsymbol{r}} = -\frac{\beta\cos(\theta)}{\alpha\, r}\hat{\boldsymbol{\theta}}, \quad \frac{\partial}{\partial z}\hat{\boldsymbol{\theta}} = \frac{\beta\cos(\theta)}{\alpha\, r}\hat{\boldsymbol{r}}. \tag{33}$$

As was stated previously, the gradient of any scalar variable such as the pressure (with $\partial p/\partial\zeta = 0$ assumed) is given using (21), and the divergence of a vector is given as in (23). Similarly, using the del operator shown in (20), the curl of a vector $\boldsymbol{u} = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{z}}$ is seen to be

$$\nabla\times\boldsymbol{u} = \hat{\boldsymbol{r}}\left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{\partial v}{\partial z} + \frac{\beta\cos(\theta)}{\alpha r}u\right) + \hat{\boldsymbol{\theta}}\left(-\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} + \frac{\beta\cos(\theta)}{\alpha r}v\right) + \hat{\boldsymbol{z}}\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{\partial u}{\partial\theta}\right), \tag{34}$$

and the gradient acting on the velocity vector $\boldsymbol{u}$ is given by

$$\nabla\boldsymbol{u} = \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\frac{\partial u}{\partial r} + \hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}}\frac{\partial v}{\partial r} + \hat{\boldsymbol{r}}\hat{\boldsymbol{z}}\frac{\partial w}{\partial r} + \frac{1}{r}\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}\left[\frac{\partial u}{\partial\theta} - v\right] + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\left[\frac{\partial v}{\partial\theta} + u\right] + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}}\frac{\partial w}{\partial\theta}\right) \tag{35}$$

$$+\hat{\boldsymbol{z}}\hat{\boldsymbol{r}}\left[\frac{\partial u}{\partial z} + \frac{\beta\cos(\theta)}{\alpha\, r}v\right] + \hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}}\left[\frac{\partial v}{\partial z} - \frac{\beta\cos(\theta)}{\alpha\, r}u\right] + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\frac{\partial w}{\partial z}. \tag{36}$$

Here we have described a second-order tensor in terms of dyadic products, for instance $\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}}^T$ [1]. To determine tensor derivatives we first note that

$$\partial_\theta(\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}) = -\partial_\theta(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}, \quad \partial_\theta(\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}}) = \partial_\theta(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}) = \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}, \tag{37}$$

$$\partial_\theta(\hat{\boldsymbol{r}}\hat{\boldsymbol{z}}) = \hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}}, \quad \partial_\theta(\hat{\boldsymbol{z}}\hat{\boldsymbol{r}}) = \hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}}, \quad \partial_\theta(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}}) = -\hat{\boldsymbol{r}}\hat{\boldsymbol{z}}, \quad \partial_\theta(\hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}}) = -\hat{\boldsymbol{z}}\hat{\boldsymbol{r}}, \tag{38}$$

$$\frac{\partial}{\partial z}(\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}) = -\frac{\partial}{\partial z}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}) = -\frac{\beta\cos(\theta)}{\alpha\, r}(\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}), \tag{39}$$

$$\frac{\partial}{\partial z}(\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}}) = \frac{\partial}{\partial z}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}) = -\frac{\beta\cos(\theta)}{\alpha\, r}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}), \tag{40}$$

$$\frac{\partial}{\partial z}(\hat{\boldsymbol{r}}\hat{\boldsymbol{z}}) = -\frac{\beta\cos(\theta)}{\alpha\, r}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}}, \quad \frac{\partial}{\partial z}(\hat{\boldsymbol{z}}\hat{\boldsymbol{r}}) = -\frac{\beta\cos(\theta)}{\alpha\, r}\hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}}, \tag{41}$$

$$\frac{\partial}{\partial z}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}}) = \frac{\beta\cos(\theta)}{\alpha\, r}\hat{\boldsymbol{r}}\hat{\boldsymbol{z}}, \quad \frac{\partial}{\partial z}(\hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}}) = \frac{\beta\cos(\theta)}{\alpha\, r}\hat{\boldsymbol{z}}\hat{\boldsymbol{r}}. \tag{42}$$

Writing a two dimensional tensor $\boldsymbol{A}$ (for example, the rate-of-strain tensor $\boldsymbol{A} = \dot{\boldsymbol{\gamma}}$) as

$$\boldsymbol{A} = A_{rr}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + A_{r\theta}\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} + A_{rz}\hat{\boldsymbol{r}}\hat{\boldsymbol{z}} + A_{\theta r}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}} + A_{\theta\theta}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + A_{\theta z}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}} + A_{zr}\hat{\boldsymbol{z}}\hat{\boldsymbol{r}} + A_{z\theta}\hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}} + A_{zz}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}, \tag{43}$$

then the divergence of $\boldsymbol{A}$ may be written as

$$\nabla \cdot \boldsymbol{A} = \left(\hat{\boldsymbol{r}}\partial_r + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) \cdot \boldsymbol{A}, \tag{44}$$

leading to the following components of the divergence in the locally orthogonal basis $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}})$:

$$\hat{\boldsymbol{r}} \cdot [\nabla \cdot \boldsymbol{A}] = \partial_r(A_{rr}) + \frac{1}{r}\partial_\theta(A_{\theta r}) + \frac{\partial}{\partial z}(A_{zr}) + \frac{1}{r}(A_{rr} - A_{\theta\theta}) + \frac{\beta\cos(\theta)}{\alpha\, r}A_{z\theta}, \tag{45}$$

$$\hat{\boldsymbol{\theta}} \cdot [\nabla \cdot \boldsymbol{A}] = \partial_r(A_{r\theta}) + \frac{1}{r}\partial_\theta(A_{\theta\theta}) + \frac{\partial}{\partial z}(A_{z\theta}) + \frac{1}{r}(A_{r\theta} + A_{\theta r}) - \frac{\beta\cos(\theta)}{\alpha\, r}A_{zr}, \tag{46}$$

$$\hat{\boldsymbol{z}} \cdot [\nabla \cdot \boldsymbol{A}] = \partial_r(A_{rz}) + \frac{1}{r}\partial_\theta(A_{\theta z}) + \frac{\partial}{\partial z}(A_{zz}) + \frac{1}{r}A_{rz}. \tag{47}$$

Finally, since

$$\boldsymbol{u} \cdot \nabla = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}, \tag{48}$$

we have (using dyadic notation),

$$\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{rr} + \Psi(A_{r\theta} + A_{\theta r}), \tag{49}$$

$$\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{r\theta} - \Psi(A_{rr} - A_{\theta\theta}), \tag{50}$$

$$\hat{\boldsymbol{r}}\hat{\boldsymbol{z}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{rz} + \Psi A_{\theta z}, \tag{51}$$

$$\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{\theta r} - \Psi(A_{rr} - A_{\theta\theta}), \tag{52}$$

$$\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{\theta\theta} - \Psi(A_{r\theta} + A_{\theta r}), \tag{53}$$

$$\hat{\boldsymbol{\theta}}\hat{\boldsymbol{z}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{\theta z} - \Psi A_{rz}, \tag{54}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{r}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{zr} + \Psi A_{z\theta}, \tag{55}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{\theta}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{z\theta} - \Psi A_{zr}, \tag{56}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \cdot [\boldsymbol{u} \cdot \nabla\boldsymbol{A}] = \boldsymbol{u} \cdot \nabla A_{zz}, \tag{57}$$

where

$$\Psi = \left[\frac{\beta\cos(\theta)}{\alpha\, r} w - \frac{v}{r}\right]. \tag{58}$$

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